HW 03

Elizabeth Hunt

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1 Question One

1.1 Three Terms

$$
\begin{aligned}
Si_3(x) &= \int_0^x \frac{s - \frac{s^3}{3!} + \frac{s^5}{5!}}{s} dx \\
&= x - \frac{x^3}{(3!)(3)} + \frac{x^5}{(5!)(5)}
\end{aligned}
$$

1.2 Five Terms

$$
\begin{aligned}
Si_3(x) &= \int_0^x \frac{s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \frac{s^9}{9!}}{s} dx \\
&= x - \frac{x^3}{(3!)(3)} + \frac{x^5}{(5!)(5)} - \frac{x^7}{(7!)(7)} + \frac{s^9}{(9!)(9)}
\end{aligned}
$$

1.3 Ten Terms

$$
\begin{aligned}
Si_{10}(x) &= \int_0^x \frac{s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \frac{s^9}{9!} - \frac{s^{11}}{11!} + \frac{s^{13}}{13!} - \frac{s^{15}}{15!} + \frac{s^{17}}{17!} - \frac{s^{19}}{19!}}{s} ds \\
&= x - \frac{x^3}{(3!)(3)} + \frac{x^5}{(5!)(5)} - \frac{x^7}{(7!)(7)} + \frac{s^9}{(9!)(9)} - \frac{s^{11}}{(11!)(11)} + \frac{s^{13}}{(13!)(13)} - \frac{s^{15}}{(15!)(15)} \\
&+ \frac{s^{17}}{(17!)(17)} - \frac{s^{19}}{(19!)(19)}
\end{aligned}
$$

2 Question Three

For the second term in the difference quotient, we can expand the taylor series centered at x=a:

$$
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots
$$

Which we substitute into the difference quotient:

$$
\frac{f(a) - f(a-h)}{h} = \frac{f(a) - (f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots)}{h}
$$

And subs. $x = a - h$:

$$
\begin{aligned}
\frac{f(a) - (f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots)}{h} &= -f'(a)(-1) + -\frac{1}{2}f''(a)h \\
&= f'(a) - \frac{1}{2}f''(a)h + \cdots
\end{aligned}
$$

Which we now plug into the initial e_{abs}:

$$
\begin{aligned}
e_{\text{abs}} &= |f'(a) - \frac{f(a) - f(a-h)}{h}| \\
&= |f'(a) - (f'(a) + -\frac{f''(a)}{2}h + \cdots)| \\
&= |-\frac{1}{2}f''(a)h + \cdots|
\end{aligned}
$$

With the Taylor Remainder theorem we can absorb the series following the second term:

$$
e_{\text{abs}} = |-\frac{1}{2}f''(a)h + \cdots| = |\frac{1}{2}f''(\xi)h| \leq Ch
$$

Thus our error is bounded linearly with h.

3 Question Four

For the first term in the difference quotient we know, from the given notes,

$$
f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{1}{6}f'''(a)(h^3)
$$

And from some of the work in Question Three,

$$
f(a-h) = f(a) + f'(a)(-h) + \frac{1}{2}f''(a)(-h)^2 + \frac{1}{6}f'''(a)(-h^3)
$$

We can substitute immediately into $e_{\text{abs}} = |f'(a) - (\frac{f(a+h) - f(a-h)}{2h})|$:

$$
\begin{aligned}
e_{\text{abs}} &= |f'(a) - \frac{1}{2h}((f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots) - (f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots))| \\
&= |f'(a) - \frac{1}{2h}(2f'(a)h + \frac{1}{6}f'''(a)h^3 + \cdots)| \\
&= |f'(a) - f'(a) - \frac{1}{12}f'''(a)h^2 + \cdots| \\
&= |-\frac{1}{12}f'''(a)h^2 + \cdots|
\end{aligned}
$$

Finally, with the Taylor Remainder theorem we can absorb the series following the third term:

$$
e_{\text{abs}} = |-\frac{1}{12}f'''(\xi)h^2| = |\frac{1}{12}f'''(\xi)h^2| \leq Ch^2
$$

Meaning that as h scales linearly, our error is bounded by h^2 as opposed to linearly as in Question Three.

4 Question Six

4.1 A

```
(load "../lizfcm.asd")
(ql:quickload :lizfcm)

(defun f (x)
  (/ (- x 1) (+ x 1)))

(defun fprime (x)
  (/ 2 (expt (+ x 1) 2)))

(let ((domain-values (loop for a from 0 to 2
                           append
                           (loop for i from 0 to 9
                                 for h = (/ 1.0 (expt 2 i))
                                 collect (list a h)))))
  (lizfcm.utils:table (:headers '("a" "h" "f'" "\\approx f'" "e_{\\text{abs}}")
                       :domain-order (a h)
                       :domain-values domain-values)
    (fprime a)
    (lizfcm.approx:fwd-derivative-at 'f a h)
    (abs (- (fprime a)
            (lizfcm.approx:fwd-derivative-at 'f a h)))))
```

5 Question Nine

5.1 C

```
(load "../lizfcm.asd")
(ql:quickload :lizfcm)

(defun factorial (n)
  (if (= n 0)
      1
      (* n (factorial (- n 1)))))

(defun taylor-term (n x)
  (/ (* (expt (- 1) n)
        (expt x (+ (* 2 n) 1)))
     (* (factorial n)
        (+ (* 2 n) 1))))

(defun f (x &optional (max-iterations 30))
  (let ((sum 0.0))
    (dotimes (n max-iterations)
      (setq sum (+ sum (taylor-term n x))))
    (* sum (/ 2 (sqrt pi)))))

(defun fprime (x)
  (* (/ 2 (sqrt pi)) (exp (- 0 (* x x)))))
```

```
(let ((domain-values (loop for a from 0 to 1
                           append
                           (loop for i from 0 to 9
                                 for h = (/ 1.0 (expt 2 i))
                                 collect (list a h)))))
  (lizfcm.utils:table (:headers '("a" "h" "f'" "\\approx f'" "e_{\\text{abs}}")
                       :domain-order (a h)
                       :domain-values domain-values)
    (fprime a)
    (lizfcm.approx:central-derivative-at 'f a h)
    (abs (- (fprime a)
            (lizfcm.approx:central-derivative-at 'f a h)))))
```

a	h	f'	$\approx$ f'	e_{abs}
0	1.0	1.1283791670955126d0	0.8427006725464232d0	0.28567849454908933d0
0	0.5	1.1283791670955126d0	1.0409997446922075d0	0.0873794224033051d0
0	0.25	1.1283791670955126d0	1.1053055663206806d0	0.023073600774832004d0
0	0.125	1.1283791670955126d0	1.122529655394656d0	0.005849511700856569d0
0	0.0625	1.1283791670955126d0	1.1269116944798618d0	0.0014674726156507223d0
0	0.03125	1.1283791670955126d0	1.1280120131008824d0	3.6715399463016496d-4
0	0.015625	1.1283791670955126d0	1.1282873617826952d0	9.180531281738347d-5
0	0.0078125	1.1283791670955126d0	1.128356232581468d0	2.293451404455915d-5
0	0.00390625	1.1283791670955126d0	1.1283734502811613d0	5.71681435124205d-6
0	0.001953125	1.1283791670955126d0	1.1283777547060847d0	1.4123894278572635d-6
1	1.0	0.41510750774498784d0	0.4976611317561498d0	0.08255362401116195d0
1	0.5	0.41510750774498784d0	0.44560523266293384d0	0.0304977249179460d0
1	0.25	0.41510750774498784d0	0.4234889628937013d0	0.008381455148713468d0
1	0.125	0.41510750774498784d0	0.41725265825950153d0	0.002145150514513694d0
1	0.0625	0.41510750774498784d0	0.41564710776310854d0	5.396000181207006d-4
1	0.03125	0.41510750774498784d0	0.4152414157140871d0	1.3390796909928948d-4
1	0.015625	0.41510750774498784d0	0.41514241394084905d0	3.490619586121735d-5
1	0.0078125	0.41510750774498784d0	0.41510582632900395d0	1.6814159838896003d-6
1	0.00390625	0.41510750774498784d0	0.415092913054238d0	1.4594690749825112d-5
1	0.001953125	0.41510750774498784d0	0.4150670865046777d0	4.0421240310117845d-5

6 Question Twelve

First we'll place a bound on h; looking at a graph of f it's pretty obvious from the asymptotes that we don't want to go much further than $|h| = 2 - \frac{pi}{2}$.
Following similar reasoning as Question Four, we can determine an optimal h by computing e_{abs} for the central difference, but now including a roundoff error for each time we run f such that $|f_{\text{machine}}(x) - f(x)| \leq \epsilon_{\text{dblprec}}$ (we'll use double precision numbers, from HW 2 we know $\epsilon_{\text{dblprec}} \approx 2.22045(10^{-16})$).
We'll just assume $|f_{\text{machine}}(x) - f(x)| = \epsilon_{\text{dblprec}}$ so our new difference quotient becomes:

$$
\begin{aligned}
e_{\text{abs}} &= |f'(a) - (\frac{f(a+h) - f(a-h) + 2\epsilon_{\text{dblprec}}}{2h})| \\
&= |\frac{1}{12}f'''(\xi)h^2 + \frac{\epsilon_{\text{dblprec}}}{h}|
\end{aligned}
$$

Because we bounded our $|h| = 2 - \frac{pi}{2}$ we'll find the maximum value of f''' between $a - (2 - \frac{\pi}{2})$ and $a - (2 - \frac{\pi}{3})$. Using desmos I found this to be -2.
Thus, $e_{\text{abs}} \leq \frac{1}{6}h^2 + \frac{\epsilon_{\text{dblprec}}}{h}$. Finding the derivative:

$$e' = \frac{1}{3}h - \frac{\epsilon_{\text{dblprec}}}{h^2}$$

And solving at $e' = 0$:

$$\frac{1}{3}h = \frac{\epsilon_{\text{dblprec}}}{h^2} \Rightarrow h^3 = 3\epsilon_{\text{dblprec}} \Rightarrow h = (3\epsilon_{\text{dblprec}})^{1/3}$$

Which is $\approx (3(2.22045(10^{-16}))^{\frac{1}{3}} \approx 8.733510^{-6}$.